Solving Equations in a Technological Environment

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What is it that a rich society owes to its citizens? Not, I think, warehousing under the pretext that they are in “ninth-grade algebra or learning Macbeth,” but something far deeper, more valuable, more personal, more meaningful. When a student is bored, I do not think it is necessarily a failure of the student, nor of the teacher, but more often a failure of the curriculum. (Davis 1994, 318)

Over the past few years, a call has been made for a dramatic shift in the algebra curriculum away from an emphasis on rote exercises and manipulations of symbolic expressions. Traditionally, solving equations as an analytic, noncontextual task monopolized algebraic manipulations and was often considered to be subject to rote learning. Technology, which generates numerical solutions mainly through the use of graphical analysis, allows us to split the traditional meaning of “solving an equation” into two parts: manipulating equations and seeing solutions. The two are no longer dependent.

Once symbolic manipulations are disconnected from the process of reaching a solution, what is the merit of carrying out such manipulations? Is solving equations with the use of technology any more meaningful than solving by rote manipulations?

We seek answers to the foregoing questions in work with algebra students, with whom we are testing an innovative algebra course. An algebra course similar to first- and second-year algebra in schools in the United States is compulsory in grades 7–9 in Israel. The algebra course we describe, adopted by some middle schools, grades 7–9, in Israel, is part of a curriculum developed by the Centre for Educational Technology, Tel-Aviv. It is an alternative course organized around the concept of function. It is based on intensive use of technology and supports an active learning environment that we call “guided inquiry into visual mathematics.” The class whose work is presented here was taught by coauthor Shoshana Gilead, who also compiled and processed the assessment tasks as part of the routine assessment in her class. The assessment sequence was also part of our curriculum-development efforts and was used to gauge the students’ “mathematical power,” as defined by the NCTM’s Assessment Standards for School Mathematics (1995, 89). In particular, we looked at students’ use of familiar representations and procedures to reason logically and to solve and communicate about unfamiliar tasks. The first task presented here followed a unit that dealt with manipulations on expressions that was supported by the software package The Function Supposer: Symbols and Graphs (Schwartz and Yerushalmy 1993). The later tasks assess students’ actions after they have learned to solve equations. The work on
the tasks was carried out in class individually but without the use of software.

**Students’ Solutions Prior to Instruction**

Find the value of $x$ for which the following equation holds:

$$2(x - 4) = -2x + 6$$

How do eighth graders, whose first few months of algebra consisted of describing and analyzing processes as functions in various representations and who had not yet studied any methods for solving equations, approach this elementary task? The first two examples are by students who, as with most of the students, restated the equation as a comparison of two functions and evaluated the values in a table. All graphs, numbers, and symbols are students’ figures with the text translated from Hebrew. Eran started (fig. 1) by evaluating negative and positive integral inputs for both functions. This process stopped once Eran identified a change in the relationship between the values in the two columns. He pointed toward the interval between 3 and 4 as surrounding the solution, guessed a value, substituted, and verified. A similar tactic of narrowing down toward an appropriate interval for the solution, using a table of values, is presented in the student’s work shown in figure 2. Although this “zoom in” table is the only table that appears in the documentation, the student explained in writing the previous stages of her solution: “I created something like a table of values until the results became very close. I checked the differences. I worked with a calculator.”

The next two works are by students who used graphs in addition to a table of values. In the first example (fig. 3), Eli did not arrive at a solution; instead, he outlined a way to find the answer. He sketched two lines and constructed two short tables. However, the observed domain did not include any outputs common to the two functions. His attempts to find a solution by computations of averages, documented beneath the tables, failed, so he then suggested that in general one has to look for the intersection point to solve the equation.

Whereas Eli’s graph was sketched as a direct description of the symbolic expression, by assumptions about the sign of the slope and the intersections with the axes, Benny’s graphs (fig. 4) were plotted from the table. Benny plotted the points in the table, connected the lines, read the intersection value and estimated it as 3.5, returned to the table and evaluated the two functions, and arrived at the solution. These four examples demonstrate the type of work observed in general: students took the approach of systematic guessing, relying
on intuitive numerical-analysis strategies to narrow down the search interval, with no use of symbolic manipulations.

**Where to Next?**

Having seen this level of mathematical sophistication, we confronted a serious dilemma: if this preactivity is typical, then how should we approach the topic of solving equations? What would be the role of, and rationale for, symbolic manipulations? How would fluency and automaticity with manipulations, which the students still lack, contribute to or counteract the extensive communication already going on in our algebra classroom?

Surprisingly, the dilemma was partially solved by the students themselves. When we discussed their numerical-analysis methods in class, they made clear that even though they always know how to start the solution process by comparing expressions, they expected that guessing would sometimes be hard to use for a few reasons. As Shira indicated, the discrete nature of the values table might be misleading:

> Identifying all the solutions by using the table, we have to use all inputs; but this is a very awkward method, and there is no way that we can solve the equation this way because we have to substitute all values.

For her, graphs may serve this purpose better than tables. But others suspected that although reading from the graph has the advantage of allowing one to watch the function on a continuous domain, it does not necessarily help to identify the best interval for the search for solutions, particularly for nonlinear functions.

Generally, students expressed the need for a more efficient and more accurate method to “solve the equation.” Efficient guessing can be achieved using technology—should technology be a constant prosthesis to replace symbolic manipulations? Accurate solutions can be achieved analytically only for the few equations that have such procedures—is that a reason to make the practice of linear and quadratic algorithms a central subject of learning? Although we were pleased by the motivated students who, despite finding a solution, still looked for improved methods, these unanswered questions promoted further exploration of the meaning of manipulations on equations.

We started by introducing one possible procedure for solving equations: transforming the two functions into a single difference function. Such a procedure requires understanding binary operation between functions and identifying equivalence between \( f(x) = g(x) \) and \( h(x) = f(x) - g(x) = 0 \). The investigation of this equivalence is supported by the software, which plots the graph and the numerical values of the difference function as part of a small set of numerical-analysis tools. Students had already used the difference function while simplifying expressions as a way of observing the similarity between the manipulated expressions, using the software mainly to plot the difference function and to verify its being the constant zero function. Thus the difference function acted as a feedback mechanism to check the simplified expressions. Here the difference function \( b(x) \) is the object to act on to simplify the equation and to inspect a solution. This role of the difference function was a subject for class discussions and was then formalized into a solving procedure.

**Students’ Solutions after Instruction on the Difference Function**

Our main concern was whether we introduced an automaticity by teaching the single symbolic procedure, one that might replace students’ intuitive numerical-analysis methods. We explored this question through students’ work on the following problem:

> Solve the following equation. Show each step of your solution.

\[ 13 - 2x = 2(x - 6) \]
For which values of $x$ does the following inequality exist?

$$13 - 2x > 2(x - 6)$$

This problem has a nonintegral solution that makes guessing harder. A procedure for solving inequalities had never been formally introduced, and this particular inequality involved an additional complexity, usually reported as a source of mistakes: the given inequality reads “larger” ($>$) but the result reads “smaller” ($<$), that is, $x < 6 1/4$. We followed students’ written work that, again, was carried out without the use of graphing or symbolic technology. In the first example (fig. 5), the equation is turned into what Eli calls a “joined function” that was then simplified to look for a solution, and the inequality was treated graphically.

Gill (fig. 6) followed a similar process, but instead of graphing the two functions, he studied the inequality by testing values in the neighborhood of the solution. Whereas Eli and Gill used the equation’s solution only to identify a domain around which to explore the inequality by numerical methods, the next work (fig. 7) is an example of how the difference function turns out to be a tool to solve the inequality.

For most students, the difference procedure served as a way to manipulate the equation and arrive at a solution. They operated with some automaticity in solving the equation. This approach is a dramatic shift from the use of comparison and numerical analysis that dominates their earlier work. Obviously, this change was an immediate reaction to the latest teaching. But more than a procedure is reflected in the work we have examined. Although all the students seem to feel comfortable with the manipulations and procedures, they constantly attempt to make sense of the procedure, to explain why it is worth pursuing. Frequently another representation that follows the symbolic manipulations is used, and choices are involved in solving unfamiliar tasks, such as the inequality presented previously.
Where to Next?

In our search for meaningful manipulations, we turned to less trivial equations, ones for which the use of procedures may still leave one puzzled about the meaning of the solution. Here is an example that we used in one of the classes when computers were available.

Solve the equation \( 4(x + 1) = 4x + 1 \).

Automatically, students operated with the learned procedure and arrived at \( 3 = 0 \). This result is surprising for someone who views this procedure as one that leads to the evaluation of a missing variable \( x \). One by one, the students turned to use the computers, graphed the two functions, and looked at what appeared to be two parallel lines and at the constant-difference function. Then the discussion focused on the connection between the two representations of the equation and on the meaning of the solution. Such use of the software was usually needed for students' first exploration of a new complexity, and then similar considerations were used while working without the graphing technology, such as in the assessment task seen in figure 8. This task presents a linear equation with an empty solution set. Benny used the difference procedure to reach the solution, but when the difference reduced to a number, he turned to demonstrate the result graphically.

The visual considerations turned out to be more controversial, however, when nonlinear equations were involved. For example, when asked to solve the quadratic \( x^2 + 5x = 10 + x^2 \) with no given procedure, students turned to the software and graphed both functions. (See fig. 9.) The analysis led to a linear difference function, which suggested a single solution to the quadratic. This result was not accepted by many, who thought that continuous rescaling of the given graphs might lead to a second, still invisible, intersection point. In the ensuing discussion we resolved the disagreement by symbolic manipulations that verified the single-solution visual conclusion. This teaching unit never approached procedures for solving quadratic equations, and solutions were read and analyzed using the software. To further investigate whether students were using various representations intelligently without the immediate graphical and numerical aid of the software, we gave the following problem.

Jo was asked to determine the number of solutions for the following equation:

\[ x^2 + 4 = x + 24 \]

1. Using the table of values he decided that the equation has one solution. Explain how he arrived at his conclusion.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f(x) - g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>20</td>
<td>28</td>
<td>-8</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
<td>29</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>53</td>
<td>31</td>
<td>22</td>
</tr>
<tr>
<td>8</td>
<td>66</td>
<td>30</td>
<td>36</td>
</tr>
</tbody>
</table>

2. Was Jo right? Can you suggest how he could check his answer? Explain.

The problem, which we usually present to students who have already practiced solving quadratic equations, yielded a crop of answers from our students who had not solved quadratic equations. Most of their suggested strategies blended numerical and graphical analysis with algebraic symbols and sometimes even included reasoning about procedures.
In the first example (fig. 10), Benny criticized the information in the table. He claimed that his analysis of the graph, sketched to reflect the properties of the function from the expressions rather than from points, offered another conclusion. Benny observed the relations between the two graphs by evaluating and comparing the constant term $b$, 4 in one function and 24 in the other. He made a careful analysis and looked for numerically and graphically convincing arguments.

Eli (see fig. 11) analyzed the difference function and its zeros using the table. His objection to Jo’s method was the use of a misleading domain interval that did not include negative values. He offered two alternative approaches, one with which he was familiar—the number of solutions could be determined by evaluating the graphs—and another about which he could only conjecture—the exact solution values could result from a solving procedure that had never been presented to him but should exist.

In general, the most frequent strategy was to use graphs, not point-plotting graphs but rather iconic graphs that can be manipulated. The graphs reveal the type of function, its relation to the axes, and its relative location to the compared function. The parabola icon was relocated to the correct position, and the lines, to intersect the axes in appropriate orientations. This type of manipulation leads to a more efficient search for the number of solutions and their values. The example in figure 12 is the work of a student who presented a repertoire of approaches to the question.

In part (a) of his answer (fig. 12), Eran started with an analysis of the given equation using an iconic graph to demonstrate the possibility of more than one solution. In part (b), he then turned the icon into a graph that described the equation more accurately. In part (c), Eran arrived at a simplified equation, but here, unlike in the linear equation, the quadratic-difference expression, which he did not know how to factor or manipulate, is the subject of a guess rather than of arithmetic reverse operations.

The symbolic manipulations demonstrated here were an ongoing combination of analytic and numerical methods, each chosen by students for a reason and to meet a need. Comparisons of two functions provide a dynamic image for solving
equations. Students' work completely contradicts the assumption that inspecting a solution while skipping the analytic process for arriving at a solution may leave “solving” a meaningless action that cannot be controlled by the learner.

The segment of teaching we describe here dealt only with the use of the difference procedure and not with other kinds of manipulations that produce equivalent equations. This difference equation allowed an easier inspection of the result using simple arithmetic operations, and in fact students named it “the simplified equation” or the “simplest function,” “the joint function,” and the “concatenated form of the equation.” A similar procedure could also lend meaningful insight and be helpful in solving quadratic equations. However, a complete study of equations obviously requires an understanding of manipulations on both sides of the equation. Should that requirement suggest that manipulations of equivalent equations be introduced only in an advanced stage of the learning when such a clear motive exists (e.g., for the study of multivariables equations, which is more often recognized as changing the subject of the formula)? Or should we use linear equations as an opportunity to prepare for this advanced study and teach operations on equations even without immediate need or motive? We try to learn more about this dilemma by using graphing technology to teach the manipulations of equivalent equations in a single variable and studying its effects on the manipulations of equations in two variables.

Finally, we would like to return to the main concern of this work as raised in the introduction: what are possible meaningful manipulations of algebraic equations when dealing with analytic tasks and using multirepresentational technology? It is hard to imagine how such a curriculum and inquiry could be carried out without technology. However, we feel that the technology was not at all a sufficient condition to promote nonrote learning. The nature of the tasks, which the technology made interesting to explore, changed the forms of the study of manipulations. We could identify two important components in the tasks we used: We asked students to invent procedures, such as the one for the inequality, by looking at a variety of examples, conjecturing and justifying them using symbolic manipulations when available. Furthermore, we tried to pose problems that cannot be answered directly by reading the screen but rather that require interpretations of the visible outcomes. The role of the technology we used was to furnish examples and representations to analyze and from which to reason. But more important, we think, to engage our students with meaningful manipulations, our roles as a teacher and as a curriculum developer changed: we could no longer follow the preplanned sequence and activities. We had to view and explore where our students were daily, what topics needed to be practiced without technology to achieve some automaticity, and what topics should be posed as a subject for exploration with technology. Such an experimentation of meaning is part of what we want our students always to participate in, as Davis suggested: “something far deeper, more valuable, more personal, more meaningful.”

Reference

