Intuition, Schema, and Problem Solving

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Intuition and schema may seem like contradictory concepts. In this work I present the relationship between these two concepts, and show their influence in the process of mathematical problem solving in primary school.

Intuition

The concept of intuition has been discussed by philosophers, cognitive psychologists, and mathematics educators. Descartes (1637/2001), in his *Discourse on Method, Optics, Geometry, and Meteorology*, applied the Cartesian method (attained through the use of pure mathematics) to several fields of human endeavor. He wrote that physics could not be shown to be an organic part of human wisdom unless and until its principles could be truly known. This is the fundamental place of intuition in the Cartesian system: the clear and distinct apprehension of proposition seen to be self-evidently true. Descartes wrote, "Intuition occurs either after or concomitantly with analysis" (p. xii) and continued, "Methods consist of a set of rules or procedures for using the natural capacities and operations of the mind correctly; there are only two operations: intuition and deduction". Descartes wrote that intuition and deduction are so closely related that it is perhaps difficult to conceive of them as two operations at all, since deduction is distinguishable from intuition only by the fact that it involves "a certain movement or succession" and that "present evidence is not necessary, as it is to intuition … the first principles of knowledge are known by intuition, and the conclusions from them are known by deduction". Olscamp, in his introduction to Descartes' book, presented the process of "analysis-intuition-deduction" as a way to achieve certain knowledge. Olscamp also related to Plato's work and said that in using analysis we treat our starting assumptions as hypotheses until we intuitively grasp their truth. Plato related to intuition as continuing analysis and preceding synthesis. He described the entire process of "analysis-intuition-synthesis" through a metaphor of "the stepped arch, in which intuition was the copping stone that held the sides of the arch, analysis and synthesis, together."(p.16).

Kant (1787/1855), in his book *Critique of Pure Reason* relates to intuition as a way in which objects are directly grasped, "possibility of a priori cognition" (p. 92). Fischbein (1987) distinguished between two kinds of intuitions: "affirmatory intuitions" and "anticipatory intuitions". Affirmatory intuitions are direct and self-evident cognition without the need for checking and proving.
They are associated with a feeling of certitude or intrinsic conviction, and exert a coercive effect on the individual's reasoning strategies to reject alternative solutions or hypotheses. Intuitions have the capacity to extrapolate beyond the empirical support (even if the idea is incorrect) and involve global cognition (in contradiction to logical or analytical). Fischbein (1999) stated that intuition is, generally, the effect of a compression if a structural schema lies behind this cognition. Fischbein quoted Thurston (1990) who said:

Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. (p. 887).

Anticipatory intuitions are part of the process of problem solving. They begin with a meaningful grasp of the problem: distinguishing between the given information and the question. The solver then has to search in his mind for different strategies of problem solving that he already knows, to bridge the gap between the given and what is required. Finally, to solve the problem the solver must reach a schema.

**Schema**

Philosophers and psychologists consider the notion of schemata in different ways. The term *schema* is used as a means of perceiving the world, as an innate logical development, and as a pattern of action.

Kant (1787/1855), like some Greek philosophers, used the term schemata for describing objects and concepts. He described schema as the link between perceiving real world objects and categories of pure understanding. He continued, "...the schema of the conception of understanding and the procedure of understanding … Schematism of the pure understanding… The schema is a product of imagination but distinguishable from the image” (pp. 108-109). He continued to distinguish "schema of substance", "schema of cause", schema of community", "schema of possibility", schema of reality", "schema of necessity", "schema of quality", "schema of relation", and "schema of modality and its categories". Kant concluded that schemata refer to "all possible objects following the arrangements of the categories” (p. 112). Schema, the representation of an object, does not relate directly to a specific object.

A schema is constructed according to the necessary conditions of the unity of reason – the schema of a thing in general, which is useful towards the production of the highest degree of systematic unity in the empirical exercise of reason, ….it merely indicates how, under the guidance of the idea, we ought to investigate the constitution and the relations of objects in the world of experience. (p. 411)
Rumelhart (1980) described schemata as "the building blocks of cognition". He discussed schemata while taking into account different notions, such as frames, scripts, retrieving information, organizing actions, allocating sources, and guiding the flow of processing as well as spatial and functional relationships characteristic of an object (like a chair).


Fischbein based his definition on Piaget’s notion of a schema which defines a schema not merely as a perceptual framework, but rather as a pattern of action (Fischbein & Grossman, 1997; Fischbein, 1999). Fischbein in particular believed that a schema is also a strategy for solving a certain class of problems. He stressed the behavioral aspect of a schema - a plan for action. An example of a kinetic schema of action is opening a door by its handle. Knowing that the handle must be pushed down and the door pushed in or pulled out is a schema. We rarely think about it because it is instinctive, but when we encounter an unfamiliar situation we need to construct a new schema of action. Fischbein (1999) distinguished between two types of schemata: the first "indicates a kind of condensed, simplified representation of a class of objects or events" (p. 36), and the second is "adaptive behavior of an organism …achieved by assimilation and accommodation" (p. 37). For Fischbein, a schema is a program which enables the individual to (a) record, process, control, and mentally integrate information, and (b) react meaningfully and efficiently to the environmental stimuli.

Cognitive psychologists of the 1980s considered schemata to be semantic nets expressing relations, or scripts of behavior, such as well known behavior at a birthday party or at a restaurant (Schank & Abelson, 1977; Anderson, 1980). Within the stream of cognitive psychology of the 1980s we can find the following description of what a schema is. Howard (1987) wrote, "A schema is a mental representation of some aspect of the world. It has slots that are related to each other in prescribed ways and that are filled by stimuli to create an instantiation of the schema" (p. 176).

Rumelhart and Norman (1985) characterized schemata as follows:

Schemata are data structures for representing the generic concepts stored in memory. There are schemata for generalized concepts underlying objects, situation, events, sequences of events, action and sequences of actions. …Schemata in some sense represent the stereotypes of these concepts. Roughly, schemata are like models of the outside world. To process information with the use of schema is to determine which model best fits the incoming information.
Some important features of schemata:
1. Schemata have variables;
2. Schemata can be embedded, one within another;
3. Schemata represent knowledge at all levels of abstraction;
4. Schemata represent knowledge rather than definitions;
5. Schemata are active recognition devices whose processing is aimed at the evaluation of their goodness of fit to the data being processed.

(pp.35-36)

Although a schema is composed of many details, it is not merely a pile of objects, but rather an organized collection of objects with the relations between them giving meaning to all its components. For example (Rumelhart, 1980), the components in Figure 1a seem to hardly have any meaning, yet within the schema face, they are readily and fully understood (Figure 1b).

The major functions of schemata as Rumelhart put it are: perceiving, understanding, remembering, learning, and problem solving.

**Problem solving**
1. Simple problems
1.1 Additive problems

As a child starts to describe the world with numbers he starts the development of his mathematical schemata for problem solving (Nesher et al, 1982; Greeno, 1978, 1985; Fuson, 1992). The child's first steps in counting develop the schema of describing sets of objects. As the child grows he attains new schemata, such as how a set of objects varies when objects are added to it or removed from it. He then can learn and understand mathematical language and operations, and acquire the ability to describe changes of reality in mathematical language.

A child with this limited knowledge could solve, for example, problem (no. 1): *There are 4 green apples and 3 red apples in the basket. How many apples are there in the basket?* With the schema of counting and building sets (of 4 objects and of 3 objects), the child could put them together and count them all.
However, the child will solve incorrectly if he attempts to use the above schema to answer the question (problem no. 2): *There are 7 apples in the basket, 3 of them are red and the others are green. How many green apples are there in the basket?* Intuitively, he will try to use the mathematical expression $7+3$.

The child will continue to act according to the schema of building sets and counting until he acquires a "part-part-whole" schema and an understanding of class inclusion (Piaget, 1970, 1971/1967). Acquiring new schemata is a result of development both of world experience and mathematical (logical) understanding.

With the schema of “part-part-whole” the child is able to represent set relations with a slot for an unknown quantity. The schema at this level is related to understanding the additive structure, the relation between the operations "+ " and "-" and their reversibility, with the "=" sign as denoting an equivalent relation.

The underlying schema for this level is schematically described in Figure 2. Note that the Part-Part-Whole schema is reversible and also incorporates the arithmetic additive relationship.

![Part-part-whole schema (additive schema)](image)

Figure 2. Part-part-whole schema (additive schema)

Now the child can solve problems in which there is an unknown quantity and he can use mathematical expression for finding it. The child can represent problem no. 2 in a part-part-schema (Figure 3).

![Part-part-whole schema for problem no.2](image)

Figure 3. Part-part-whole schema for problem no.2

At this stage, with the new schema the child can solve the second problem with the mathematical expression $7-3 = ?$ or $3 + ? = 7$. He understands that the red apples are part of all the apples.
For more details on the development of different stages in learning different kinds of additive problems with related schemata, in the process of building the entire additive schema, see Nesher (1982), Nesher, Greeno, and Riley (1982), Riley, Greeno, and Heller (1983), and Riley and Greeno (1988).

1.2 Multiplicative problems

Similarly to the additive schema, the multiplicative schema has three components, by which it is possible to describe all multiplicative problems. The basic schema holds two factors and a multiplicand (Figure 4).

![Figure 4. Multiplicative schema](image)

In figure 5 we present a schema for solving this problem (no. 3): *Tom distributed 50 roses equally among 5 vases. How many roses were in each vase?*

![Figure 5. Multiplicative schema for problem no.3](image)


2. Complex problems

The additive schema and the multiplicative schema are the building blocks of all the problems a child can solve with the simple four operations. (addition, subtraction, multiplication, and division). In the new schema for a more complex problem the basic schemata become "slots" in the new schema (according to Howard's (1987) terminology). Once a child can grasp easily, record, process, and control each one of the schemata (additive and multiplicative) and can mentally integrate information which belongs to the appropriate schema, the
next step is to react meaningfully and efficiently to the environmental stimuli (Fischbein 1999).

2.1 Two-step problems

Based on the analysis of two-step word problems in which one of the operations is addition or subtraction and the other operation is multiplication or division, Nesher and Hershkovitz (1994) found that all problems could be categorized into three compound schemata (Figure 6).

Any compound schema for two-step problems consists of two simple schemata. They differ by the components by which the two simple schemata are combined. In the hierarchical schema, the whole of one schema becomes part of the other schema. In the shared part schema, the two schemata share one part. In the shared whole schema, the two schemata share one whole.

Each compound schema encapsulates different problems which describe the same situation. For example we will look at a situation in which there are 45 pieces of fruit. Fifteen of them are bananas and the rest are oranges. The oranges are distributed equally between 6 plates. There are 5 oranges in each plate. The schema of the situation is represented in Figure 7.

Figure 6. Two-step problem schemata

Figure 7. Representation of a two-step problem
According to the above situation it is possible to formulate 4 different two-step problems:

a. Dina has 15 bananas and 6 plates with 5 oranges on each. How many pieces of fruit does Dina have?

b. Dina has 6 plates on each of which there are 5 oranges. The rest of the fruit are bananas. How many bananas does Dina have?

c. Dina has 45 pieces of fruit. 15 of them are bananas, and the rest are oranges. She put the oranges on plates 5 oranges on each plate. How many plates are there?

d. Dina has 45 pieces of fruit. 15 of them are bananas, and the rest are oranges. She put all the oranges on 6 plates, so that there are an equal number on each plate. How many oranges she put on each plate?

This analysis revealed both the range of difficulty of the different problems and an instructional approach for teaching children this kind of problem. It was found that children who used schemata for solving problems succeeded more than those who solved without schemata. For more details see Hershkovitz and Nesher (1996, 1998). An interesting point in the analysis is the two ways in which a schema is used, according to Fischbein (1999): (a) a kind of condensed, simplified representation of a class of objects or events, and (b) adaptive behavior. The above analysis can be also be extended to multi-step problems.

2.2 Non-routine problems

By non-routine problems we mean: problems to which a child was not exposed to at school (he does not have "ready schemata" for solving them); problems with multiple solutions; and problems that require real world knowledge and constraints. The purpose for exposing a child to such problems is to give him the opportunity to use different strategies of problem solving (Polya 1945/1957, Schoenfeld 1992), to let him build schemata for problem solving, and to use mathematics in non-mathematical contexts.


Figure 8. Non-routine problem
The problem is non-routine in different ways: it is not clear which mathematical ideas are necessary to solve the problem; no clear instruction is given as to whether the solver has to find one or more patterns, or exhaust all possible solutions. This problem was posed to children of different ages and different levels as well as to teachers and students. The solvers produced various solutions. Some related to their mathematical knowledge and some to their schema for problem solving.

In this paper we will present only Ron's protocol. Ron is an able 3rd grade student.

Ron: Do I have to use all the candies?
Teacher: No, you can use the same color more than once in the same pattern. You have to choose a pattern so that the 17th candy will be purple.
Ron placed 5 different candies in this order: blue, purple, orange, green, and red, and duplicated them.
T: How did you know?
R: 17 divided by 5 gives the pattern 3 times and the remainder is 2, so the purple has to be the second.
T: Great. Do you think there are more solutions?
R: Thought for a while… I'll try with 4 candies.
T: And?
R: The purple will be the first.
T: How?
R: 17 divided by 4 gives 4, remainder 1. The remainder is 1, so this is the purple candy.
T: Do you have an idea for "the rule" of the game?
Ron tried patterns with 3 candies and with 2 candies.
R: The place of the purple candy is the remainder of the division exercise.

Ron began with a pattern of 5 candies and intuitively found that the remainder of the division exercise gives the right answer. After some more trials with smaller patterns he became confident with his answer. His intuition for the answer is a product of some analysis (Descartes' "analysis-intuition-synthesis"). He found an answer, and did not try to extend the solution space. Maybe this was because he had acquired the schemata of multiplication and division, but not of factorization, and his schema of investigating solutions is in the beginning stages.

The same problem was solved by older students and by teachers. In our analysis we found different solutions, including different lengths of pattern and different colors in the pattern. One of the graduate students even tried to use combinatorial theory to calculate all the possible answers. We found that children used intuitively the schemata they had already learned and could even extrapolate these schemata to some extent.

Some concluding remarks

Intuitions and schemata are two complementary factors of problem solving in mathematics. It is essential to take into account the two facets of schema: a plan
for action (Piaget, 1980; Fischbein, 1999), and data structures for representing a
generic concept (Rumelhart & Norman, 1985) or mental representation
(Howard, 1987). Although both Intuitions and schemata are abstract, it is
important to encourage children to work with them.

Intuition is an important part of mathematical work. Descartes presented the
process of "analysis-intuition-deduction" as a way to achieve certain knowledge.
Feferman (2000) stressed:

_the ubiquity of intuition in the common experience of teaching and learning
mathematics, and the reasons for that: it is essential for motivation of
notions and results and to guide one's conceptions via tacit or explicit
analogies in transfer from familiar ground to unfamiliar terrain....
intuition is necessary for the understanding of mathematics. (p. 319)_

We can look at intuition and schema as a closed circuit. The more schemata a
person acquires, the more intuition he has. The educational challenge is to
enable children to develop rich mathematical schemata leading to more
intuitions for solving mathematical problems.

References

Freeman and Company, San Francisco

(Rev. ed.). (P. J. Olscamp, Trans.). Hackett Publishing. (Original work
published 1637).

Synthese, 125, 317-332.

Approach. Mathematics Education Library, Reidel Publishing Company:
Dordrecht, Holland.

Educational Studies in Mathematics, 38, 11-50.

models in solving verbal problems in multiplication and division. Journal
for Research in Mathematics Education, 16, 3-17.

reasoning. Educational Studies in Mathematics, 34, 27-47


